ON THE DAVENPORT CONSTANT AND ON THE STRUCTURE OF EXTREMAL ZERO-SUM FREE SEQUENCES

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Abstract. Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$ be a finite abelian group, $\mathsf d^*(G) = n_1 + \ldots + n_r - r$, and let $\mathsf d(G)$ denote the maximal length of a zero-sum free sequence over G. Then $\mathsf d(G) \ge \mathsf d^*(G)$, and the standing conjecture is that equality holds for $G = C_n^r$. We show that equality does not hold for $C_2 \oplus C_{2n}^r$, where $n \ge 3$ is odd and $r \ge 4$. This gives new information on the structure of extremal zero-sum free sequences over C_{2n}^r.

1. Introduction

Let G be an additively written finite abelian group, $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ its direct decomposition into cyclic groups, where $r = \mathsf r(G)$ is the rank of G and $1 < n_1 \mid \ldots \mid n_r$, and set

$$\mathsf d^*(G) = \sum_{i=1}^{r} (n_i - 1)\,, \quad \text{with} \quad \mathsf d^*(G) = 0 \quad \text{for } G \text{ trivial}\,.$$

We denote by $\mathsf d(G)$ the maximal length of a zero-sum free sequence over G. Then $\mathsf D(G) = \mathsf d(G) + 1$ is the Davenport constant of G (equivalently, $\mathsf D(G)$ is the smallest integer $\ell \in \mathbb N$ such that every sequence S over G of length $|S| \ge \ell$ has a non-trivial zero-sum subsequence). The Davenport constant has been studied since the 1960s, and it naturally occurs in various branches of combinatorics, number theory, and geometry. There is a well-known chain of inequalities

$$(*) \qquad \mathsf d^*(G) \le \mathsf d(G) \le (n_r - 1) + n_r \log \frac{|G|}{n_r}\,,$$

which obviously is an equality for cyclic groups ([14, Theorem 5.5.5]). Furthermore, equality on the left side holds for p-groups, groups of rank two and others (see [12, Sections 2.2 and 4.2] for a survey, and [3, 25, 2, 26, 7, 24, 19] for recent progress). In contrast to these results, there are only a handful of explicit families of examples showing that $\mathsf d(G) > \mathsf d^*(G)$ can happen, but the phenomenon is not understood at all. The two main conjectures regarding $\mathsf D(G)$ state that equality holds in the left side of $(*)$ for groups of rank three and for groups of the form C_n^r.

In addition to the direct problem, the associated inverse problem with respect to the Davenport constant—which asks for the structure of maximal zero-sum free sequences—has attracted considerable attention in the last decade. An easy exercise shows that a zero-sum free sequence of maximal length over a cyclic group consists of one element with multiplicity $\mathsf d(G)$. A conjecture on the structure of such sequences over groups of the form $C_n \oplus C_n$ was first stated in [8, Section 10]. After various partial results, this conjecture was settled recently: even for general groups of rank two the structure of minimal zero-sum sequences with maximal length was completely determined (see [11, 23, 20]). Apart from groups of rank two (and apart from the trivial case of elementary 2-groups) such a structural result is known only for groups of the form $C_2^2 \oplus C_{2n}$ (see [22]).

The inverse results for groups of rank two support the conjecture that $\mathsf d^*(G) = \mathsf d(G)$ holds for groups of rank three (which is outlined in [22]). Much less is known for groups of the form C_n^r. There is a covering result ([9, Theorem 6.6]), which slightly supports the conjecture that $\mathsf d^*(G) = \mathsf d(G)$ holds, and

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there is recent work by B. Girard ([16, 18]) on the order of elements occurring in zero-sum free sequences of maximal length.

In this paper, we present a series of groups of rank five, namely $G_n = C_2 \oplus C_{2n}^4$ with $n \geq 3$ odd, such that $\mathsf{d}(G_n) > \mathsf{d}^*(G_n)$ (see Theorem 3.1). This is the first series of groups for which equality in the left side of $(*)$ fails and which is somehow close to the form C_n^r (all groups known so far satisfying $\mathsf{d}^*(G) < \mathsf{d}(G)$ are quite different). Moreover, these examples shed new light on recent conjectures by B. Girard concerning the structure of extremal sequences (see Corollary 3.2 and the subsequent remark). A computer based search in the group $C_2 \oplus C_{10}^4$ was substantial for our work. This will be outlined in Section 4.

2. Preliminaries

Our notation and terminology are consistent with [10] and [14]. We briefly gather some key notions and fix the notation concerning sequences over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Throughout, all abelian groups will be written additively, and for $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements.

Let G be a finite abelian group. For a subset $A \subset G$, we set $-A = \{-a \mid a \in A\}$. An s-tuple $(e_1, \ldots, e_s)$ of elements of G is said to be independent (or more briefly, the elements $e_1, \ldots, e_s$ are said to be independent) if $e_i \neq 0$ for all $i \in [1, s]$ and, for every s-tuple $(m_1, \ldots, m_s) \in \mathbb{Z}^s$,

$$m_1 e_1 + \ldots + m_s e_s = 0 \qquad \text{implies} \qquad m_1 e_1 = \ldots = m_s e_s = 0\,.$$

An s-tuple $(e_1, \ldots, e_s)$ of elements of G is called a basis if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_s \rangle$. For a prime $p \in \mathbb{P}$, we denote by $G_p = \{g \in G \mid \operatorname{ord}(g) \text{ is a power of } p\}$ the p-primary component of G, and by $\mathsf{r}_p(G)$, the p-rank of G (which is the rank of G_p).

Let $\mathcal{F}(G)$ be the free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,, \quad \text{with} \quad \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all} \quad g \in G\,.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S *contains* g if $\mathsf{v}_g(S) > 0$. A sequence S_1 is called a *subsequence* of S if $S_1 \,|\, S$ in $\mathcal{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \leq \mathsf{v}_g(S)$ for all $g \in G$). If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)\,,$$

we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S\,, \text{ and}$$

$$\Sigma(S) = \Big\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, l] \Big\} \subset G \qquad \text{the } \textit{set of subsums} \text{ of } S\,.$$

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no non-trivial zero-sum subsequence, and
- a *minimal zero-sum sequence* if $1 \neq S$, $\sigma(S) = 0$, and every $S' | S$ with $1 \leq |S'| < |S|$ is zero-sum free.

3. The Main Theorem and its Corollary

Theorem 3.1. *Let $G = C_2^i \oplus C_{2n}^{5-i}$ with $i \in [1,4]$ and $n \ge 3$ odd. Then $\mathsf{d}(G) > \mathsf{d}^*(G)$.*

Before we start the proof of Theorem 3.1, we would like to remark that its statement easily extends to groups of higher rank. Indeed, let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$ and let $\emptyset \ne I \subset [1,r]$. If

$$\mathsf{d}\big(\oplus_{i\in I} C_{n_i}\big) > \mathsf{d}^*\big(\oplus_{i\in I} C_{n_i}\big)\,,$$

then a straightforward construction shows that $\mathsf{d}(G) > \mathsf{d}^*(G)$ (see [14, Proposition 5.1.11]). Thus the interesting groups G with $\mathsf{d}(G) > \mathsf{d}^*(G)$ are those with small rank. Recall that there is no known group G of rank three with $\mathsf{d}(G) > \mathsf{d}^*(G)$, and there is only one series of groups G of rank four such that $\mathsf{d}(G) > \mathsf{d}^*(G)$ (see [15, Theorem 3]).

Proof of Theorem 3.1. For $i \in \{3,4\}$, this follows from [15, Theorem 4], and, for $i = 2$, from [8, Theorem 3.3]. Suppose that $i = 1$ and let $(e_1, \ldots, e_5)$ be a basis of G with $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = \ldots = \mathrm{ord}(e_5) = 2n$. We define

$$\begin{aligned}
g_1 = e_1 + e_2, \quad g_2 &= e_1 + e_3, \quad g_3 = e_1 + e_4, \quad g_4 = e_1 + e_5\,,\\
g_5 = &\quad \frac{3n-1}{2}e_2 + \frac{3n+1}{2}e_3 + \frac{3n+1}{2}e_4 + \frac{3n+1}{2}e_5\,,\\
g_6 = &\quad \frac{3n-1}{2}e_2 + \frac{3n+1}{2}e_3 + \frac{3n-1}{2}e_4 + \frac{n+1}{2}e_5\,,\\
g_7 = &\quad \frac{3n+3}{2}e_2 + \frac{n+1}{2}e_3 + \frac{n-1}{2}e_4 + \frac{n+1}{2}e_5\,,\\
g_8 = &\quad \frac{n-1}{2}e_2 + \frac{n+1}{2}e_3 + \frac{3n+1}{2}e_4 + \frac{n-1}{2}e_5\,,\\
g_9 = &\quad \frac{n-1}{2}e_2 + \frac{n+1}{2}e_3 + \frac{n+1}{2}e_4 + \frac{n+1}{2}e_5\,,\\
g_{10} = &\quad \frac{3n+1}{2}e_2 + \frac{3n+1}{2}e_3 + \frac{n+1}{2}e_4 + \frac{3n+1}{2}e_5\,,\\
g_{11} = &\quad \frac{n+3}{2}e_2 + \frac{3n+1}{2}e_3 + \frac{3n+1}{2}e_4 + \frac{3n-1}{2}e_5\,,\\
g_{12} = &\; e_1 + \frac{n+1}{2}e_2 + \frac{n-1}{2}e_3 + \frac{n+1}{2}e_4 + \frac{3n+1}{2}e_5\,,
\end{aligned}$$

and assert that

$$U = g_1^{2n-2} g_2^{2n-3} g_3^{2n-2} g_4^{2n-2} g_5 g_6 g_7 g_8 g_9 g_{10} g_{11} g_{12}$$

is a minimal zero-sum sequence. Obviously, U is a zero-sum sequence of length $|U| = 8n-1 = \mathsf{d}^*(G)+2$. Thus it suffices to show that $S^* = g_{12}^{-1}U$ is zero-sum free. Let

$$S = g_1^{l_1} \cdot \ldots \cdot g_{11}^{l_{11}}$$

be a zero-sum subsequence of $g_{12}^{-1}U$, where $l_i = \mathsf{v}_{g_i}(S)$ for all $i \in [1,11]$. Thus $l_1 \in [0, 2n-2]$, $l_2 \in [0, 2n-3]$, $l_3 \in [0, 2n-2]$, $l_4 \in [0, 2n-2]$, and $l_i \in \{0,1\}$ for all $i \in [5,11]$. We have to show that $|S| = l_1 + \ldots + l_{11} = 0$.

Since $\sigma(S) = 0$, we obtain the following system of initial congruences:

$$\text{(1)} \qquad l_1 + l_2 + l_3 + l_4 \equiv 0 \mod 2\,,$$

$$\text{(2)} \qquad l_1 + \frac{3n-1}{2}l_5 + \frac{3n-1}{2}l_6 + \frac{3n+3}{2}l_7 + \frac{n-1}{2}l_8 + \frac{n-1}{2}l_9 + \frac{3n+1}{2}l_{10} + \frac{n+3}{2}l_{11} \equiv 0 \mod 2n\,,$$

$$\text{(3)} \qquad l_2 + \frac{3n+1}{2}l_5 + \frac{3n+1}{2}l_6 + \frac{n+1}{2}l_7 + \frac{n+1}{2}l_8 + \frac{n+1}{2}l_9 + \frac{3n+1}{2}l_{10} + \frac{3n+1}{2}l_{11} \equiv 0 \mod 2n\,,$$

$$\text{(4)} \qquad l_3 + \frac{3n+1}{2}l_5 + \frac{3n-1}{2}l_6 + \frac{n-1}{2}l_7 + \frac{3n+1}{2}l_8 + \frac{n+1}{2}l_9 + \frac{n+1}{2}l_{10} + \frac{3n+1}{2}l_{11} \equiv 0 \mod 2n\,,$$

$$\text{(5)} \qquad l_4 + \frac{3n+1}{2}l_5 + \frac{n+1}{2}l_6 + \frac{n+1}{2}l_7 + \frac{n-1}{2}l_8 + \frac{n+1}{2}l_9 + \frac{3n+1}{2}l_{10} + \frac{3n-1}{2}l_{11} \equiv 0 \mod 2n\,.$$

By subtracting equation (2) from (3), subtracting (4) from (3), and subtracting (5) from (3), we obtain

$$l_1 \equiv l_2 + l_5 + l_6 + l_8 + l_9 + (n-1)(l_7 + l_{11}) \mod 2n\,, \tag{6}$$
$$l_3 \equiv l_2 + l_6 + l_7 + n(l_8 + l_{10}) \mod 2n\,\text{,and} \tag{7}$$
$$l_4 \equiv l_2 + nl_6 + l_8 + l_{11} \mod 2n\,. \tag{8}$$

Next we form a congruence modulo 2, namely

$$\begin{aligned} 0 &\equiv l_1 + l_2 + l_3 + l_4 \\ &\equiv l_2 + l_5 + l_6 + l_8 + l_9 + \\ &\quad l_2 + \\ &\quad l_2 + l_6 + l_7 + l_8 + l_{10} + \\ &\quad l_2 + l_6 + l_8 + l_{11} \\ &\equiv l_5 + l_6 + l_7 + l_8 + l_9 + l_{10} + l_{11} \mod 2\,. \end{aligned}$$

Therefore we get $l_5 + l_6 + l_7 + l_8 + l_9 + l_{10} + l_{11} \in \{0, 2, 4, 6\}$. If $l_5 + l_6 + l_7 + l_8 + l_9 + l_{10} + l_{11} = 0$, then $\sigma(S) = 0$ implies immediately that $l_1 = l_2 = l_3 = l_4 = 0$ and thus $|S| = 0$. Thus we suppose that $l_5 + \ldots + l_{11} \in \{2, 4, 6\}$.

Adding (3) and (5) and inserting (8), we obtain that

$$2l_2 + (n+1)(l_5 + l_6 + l_7 + l_8 + l_9 + l_{10} + l_{11}) \equiv 0 \mod 2n\,.$$

Thus we get that either

$$l_5 + \ldots + l_{11} = 2 \quad \text{and hence} \quad l_2 = n-1$$

or

$$l_5 + \ldots + l_{11} = 4 \quad \text{and hence} \quad l_2 = n-2$$

or

$$l_5 + \ldots + l_{11} = 6 \quad \text{and hence} \quad l_2 \in \{n-3, 2n-3\}\,.$$

We distinguish these four cases.

CASE 1: $l_5 + \ldots + l_{11} = 2$ and $l_2 = n-1$.

CASE 1.1: $l_6 = 1$.

If $l_8 + l_{11} = 2$, then $l_5 = l_7 = l_9 = l_{10} = 0$, $l_1 = l_3 = 0$, and $l_4 = 1$, a contradiction to (1).

If $l_8 + l_{11} = 0$, then $l_4 = 2n-1$, a contradiction to $l_4 \in [0, 2n-2]$.

Thus we get $l_8 + l_{11} = 1$. If $l_8 = 1$, then $l_5 = l_7 = l_9 = l_{10} = l_{11} = 0$ and $l_1 = n+1$, a contradiction to (2). If $l_8 = 0$, then $l_{11} = 1$, $l_5 = l_7 = l_9 = l_{10} = 0$, and $l_1 = 2n-1$, a contradiction to $l_1 \in [0, 2n-2]$.

CASE 1.2: $l_6 = 0$.

If $l_8 + l_{10} = 2$, then $l_5 = l_7 = l_9 = l_{11} = 0$ and $l_1 = n$, a contradiction to (2).

Suppose that $l_8 + l_{10} = 0$. Then $l_4 = n-1+l_{11}$, $l_3 = n-1+l_7$, and $l_1 = (n-1)(1+l_7+l_{11}) + l_5 + l_9$. If $l_7 + l_{11} = 1$, then $l_1 = 2n-2+l_5+l_9$ and hence $l_1 = 2n-2$, a contradiction to (1). If $l_7 + l_{11} = 0$, then $l_5 = l_9 = 1$ and $l_1 = n+1$, a contradiction to (2). If $l_7 + l_{11} = 2$, then $l_5 = l_9 = 0$ and $l_1 = n-3$, a contradiction to (2).

Suppose that $l_8 + l_{10} = 1$. Then $l_3 \equiv 2n-1+l_7 \mod 2n$, which implies $l_7 = 1$ and $l_3 = 0$. Then $l_1 \equiv 2n-2+l_5+l_6+l_8+l_9 \mod 2n$, which implies $l_8 = 0$, $l_{10} = 1$, and $l_1 = 2n-2$, a contradiction to (2).

CASE 2: $l_5 + \ldots + l_{11} = 4$ and $l_2 = n-2$.

CASE 2.1: $l_6 = 1$.

If $l_8 + l_{11} = 1$, then $l_4 = 2n-1$, a contradiction to $l_4 \in [0, 2n-2]$.

Suppose that $l_8 + l_{11} = 0$. If $l_7 = 1$, then $l_1 \equiv 2n-2+l_5+l_9 \mod 2n$. Since $l_1 \in [0, 2n-2]$ and $l_5 + \ldots + l_{11} = 4$, it follows that $l_5 = l_9 = 1$ and $l_1 = 0$, a contradiction to (2). If $l_7 = 0$, then $l_5 = l_6 = l_9 = l_{10} = 1$ and $l_3 \equiv 2n-1 \mod 2n$, a contradiction to $l_3 \in [0, 2n-2]$.

Suppose that $l_8 + l_{11} = 2$. If $l_7 = 1$, then $l_5 = l_9 = l_{10} = 0$ and $l_1 = n-2$, a contradiction to (2). If $l_7 = 0$, then $l_3 \equiv n-1+n(1+l_{10}) \mod 2n$ and thus $l_{10} = 1$, $l_5 = l_9 = 0$, and $l_1 \equiv 2n-1 \mod 2n$, a contradiction to $l_1 \in [0, 2n-2]$.

CASE 2.2: $l_6 = 0$.

If $l_8 + l_{10} = 0$, then $l_5 = l_7 = l_9 = l_{11} = 1$ and $l_1 = n-2$, a contradiction to (2).

Suppose that $l_8 + l_{10} = 1$. If $l_7 = 1$, then $l_3 \equiv 2n-1 \mod 2n$, a contradiction to $l_3 \in [0, 2n-2]$. If $l_7 = 0$, then $l_5 = l_9 = l_{11} = 1$ and $l_1 \equiv 2n-1+l_8 \mod 2n$, which implies that $l_8 = 1$, $l_{10} = 0$, and $l_1 = 0$, a contradiction to (2).

Suppose that $l_8 + l_{10} = 2$. If $l_7 + l_{11} = 0$, then $l_5 = l_9 = 1$ and $l_1 = n+1$, a contradiction to (2). If $l_7 + l_{11} = 2$, then $l_5 = l_9 = 0$ and $l_1 = n-3$, a contradiction to (2). If $l_7 + l_{11} = 1$, then $l_5 + l_9 = 1$ and $l_1 \equiv 2n-1 \mod 2n$, a contradiction to $l_1 \in [0, 2n-2]$.

CASE 3: $l_5 + \ldots + l_{11} = 6$ and $l_2 = n-3$.

If $0 \in \{l_5, l_7, l_8, l_9, l_{10}\}$, then $l_4 \equiv 2n-1 \mod 2n$, a contradiction to $l_4 \in [0, 2n-2]$. If $l_6 = 0$, then $l_1 = n-2$, a contradiction to (2). If $l_{11} = 0$, then $l_1 = 0$, a contradiction to (2).

CASE 4: $l_5 + \ldots + l_{11} = 6$ and $l_2 = 2n-3$.

If $l_5 = 0$ or $l_{11} = 0$, then $l_3 \equiv 2n-1 \mod 2n$, a contradiction to $l_3 \in [0, 2n-2]$. If $l_6 = 0$, then $l_4 \equiv 2n-1 \mod 2n$, a contradiction to $l_4 \in [0, 2n-2]$. If $l_{10} = 0$, then $l_1 \equiv 2n-1 \mod 2n$, a contradiction to $l_1 \in [0, 2n-2]$. If $l_7 = 0$, then $l_1 = n$; if $l_8 = 0$, then $l_1 = 2n-2$; if $l_9 = 0$, then $l_1 = 2n-2$. All these three cases give a contradiction to (2). $\square$

In two recent papers, B. Girard states a conjecture on the structure of extremal zero-sum free sequences. We recall the required terminology.

Let $G = C_{q_1} \oplus \ldots \oplus C_{q_s}$ be the direct decomposition of the group G into cyclic groups of prime power order, where $s = \mathsf{r}^*(G) = \sum_{p \in \mathbb{P}} \mathsf{r}_p(G)$ is the total rank of G, and set

$$\mathsf{k}^*(G) = \sum_{i=1}^{s} \frac{q_i - 1}{q_i}, \quad \text{with} \quad \mathsf{k}^*(G) = 0 \text{ for } G \text{ trivial}.$$

For a sequence $S = g_1 \cdot \ldots \cdot g_l$ over G,

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)}$$

denotes its *cross number*, and

$$\mathsf{k}(G) = \max\{\mathsf{k}(U) \mid U \in \mathcal{F}(G) \text{ zero-sum free}\} \in \mathbb{Q}$$

is the *little cross number* of G. If $(e_1, \ldots, e_s)$ is a basis of G with $\operatorname{ord}(e_i) = q_i$ for all $i \in [1, s]$, then $S = \prod_{i=1}^{s} e_i^{q_i - 1}$ is zero-sum free and hence $\mathsf{k}^*(G) = \mathsf{k}(S) \le \mathsf{k}(G)$. Equality holds in particular for p-groups, and there is no known group H with $\mathsf{k}^*(H) < \mathsf{k}(H)$. We refer to [14, Chapter 5] for more information on the cross number and to [17, 13] for recent progress. Now we formulate the conjecture of B. Girard (see [16, Conjecture 1.2] and [18, Conjecture 2.1]).

Conjecture (B. Girard)**.** *If $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$ and $S \in \mathcal{F}(G)$ is zero-sum free with $|S| \ge \mathsf{d}^*(G)$, then*

$$\mathsf{k}(S) \le \sum_{i=1}^{r} \frac{n_i - 1}{n_i}.$$

The conjecture holds true for cyclic groups, p-groups (see [16, Proposition 2.3]) and for groups of rank two (this follows from the characterization of all minimal zero-sum sequences of maximal length, [23, 11]). Suppose that $G = C_n^r$. If true, the conjecture would imply that $\mathsf{d}(G) = \mathsf{d}^*(G)$ and, moreover, that all elements occurring in a zero-sum free sequence of length $\mathsf{d}^*(G)$ have maximal order n ([16, Proposition 2.1]).

Corollary 3.2. *Let $G = C_{2n}^r$ with $n \ge 3$ odd and $r \ge 5$. Then there exists a zero-sum free sequence $T \in \mathcal{F}(G)$ and an element $g \in G$ with $\operatorname{ord}(g) = n$ such that*

$$\mathsf{v}_g(T) = n-1, \ |T| = \mathsf{d}^*(G) - (n-2) \quad \textit{and} \quad \mathsf{k}(T) = r\frac{2n-1}{2n} + \frac{1}{2n}.$$

In particular, if $n = 3$ and $r = 5$, then $|T| = \mathsf{d}^*(G) - 1$, $\mathsf{k}(T) > r(2n-1)/(2n)$, and there is no zero-sum free sequence $T^* \in \mathcal{F}(G)$ such that $T^* = g_1 g_2 T'$ and $T = (g_1 + g_2)T'$, where $g_1, g_2 \in G$ and $T' \in \mathcal{F}(G)$.

Proof. Let $(e_1', e_2, \ldots, e_r)$ be a basis of G with $\mathrm{ord}(e_1') = \mathrm{ord}(e_2) = \ldots = \mathrm{ord}(e_r) = 2n$. Let $e_1 = ne_1'$ and $S^* \in \mathcal{F}(\langle e_1, e_2, e_3, e_4, e_5\rangle)$ be as constructed in the proof of Theorem 3.1. Then

$$|S^*| = 8n - 2 \quad \text{and} \quad \mathsf{k}(S^*) = \frac{|S|}{2n}\,.$$

Since $\mathrm{ord}(2e_1') = n$ and $\langle 2e_1', e_6, \ldots, e_r\rangle \cap \langle e_1, \ldots, e_5\rangle = \{0\}$, the sequence

$$T = (2e_1')^{n-1} S^* \prod_{i=6}^{r} e_i^{2n-1}$$

is zero-sum free and has the required properties.

In the case $n = 3$ and $r = 5$, we have checked numerically—by a variant of the SEA (see Algorithm 1) with reduced search depth—that there is no such sequence T^*, and the remaining assertions follow from the general case of the corollary. □

Remark 3.3. Thus, for the group $G = C_6^5$, the sequence T given in Corollary 3.2 shows that the Conjecture is sharp, in the sense that the assumption $|S| \ge \mathsf{d}^*(G)$ cannot be weakened to $|S| \ge \mathsf{d}^*(G) - 1$. But it shows much more.

Suppose that G is cyclic of order $|G| = n \ge 3$. A simple argument shows that $\mathsf{d}(G) = \mathsf{d}^*(G) = n - 1$ and every zero-sum free sequence S of length $|S| = n - 1$ has the form $S = g^{n-1}$ for some $g \in G$ with $\mathrm{ord}(g) = n$. It was a well-investigated problem in Combinatorial Number Theory to extend this structural result to shorter zero-sum free sequences. In 2007, S. Savchev and F. Chen could finally show that, for every zero-sum free sequence S of length $|S| > (n+1)/2$, there is a $g \in G$ such that $S = (n_1 g) \cdot \ldots \cdot (n_l g)$, where $l = |S| \in \mathbb{N}$, $1 = n_1 \le \ldots \le n_l$, $n_1 + \ldots + n_l = m < \mathrm{ord}(g)$ and $\Sigma(S) = \{g, 2g, \ldots, mg\}$ (see [21] and [12, Theorem 5.1.8]). Thus S is obtained by taking some factorization $(g^{n_1}) \cdot \ldots \cdot (g^{n_l}) = g^{m-1}$ of the sequence g^{m-1} and replacing each g^{n_i} by $\sigma(g^{n_i}) = n_i g$ for $i \in [1, l]$. By Corollary 3.2, such a result does not hold for C_6^5, not even for zero-sum free sequences of length $\mathsf{d}^*(G) - 1$.

4. Description of the Computational Approach

Computational methods have already been used successfully for a variety of zero-sum problems (see recent work of G. Bhowmik, Y. Edel, C. Elsholtz, I. Halupczok, J.-C. Schlage-Puchta et al. [6, 4, 5, 1]). Inspired by former work in the groups $C_2^2 \oplus C_{2n}^3$ for $n \ge 3$ odd, we found many examples of zero-sum free sequences S over $G = C_2 \oplus C_6^4$ of length $|S| = \mathsf{d}^*(G) + 1$. These were used as starting points in a computer based search in the group $C_2 \oplus C_{10}^4$, which will be explained in detail below.

The Sequence Extension Algorithm (SEA) (see Algorithm 1) uses a smart brute force approach, where the computation time is significantly reduced by algorithmic short-cuts, efficient data structures for set testing, and fast look-up tables for group operations. The program was implemented in the C/C++ programming language. Furthermore, MPI parallelization was used to enable the execution of the program on cluster computers and supercomputers with thousands of computing cores. The parallelization scheme is a simple master-slave algorithm, where the master thread partitions the outermost loop over all group elements and sends out these work items to the available pool of slave threads. In this scheduler, a dynamic policy with chunk size one is used; that is, the master thread sends out only one work item to the next slave thread available. Although this leads to some communication overhead between the master and the slave threads, it is quite reasonable as the necessary computation time for one work item can vary by a factor of more than 25000, i.e., from less than a second up to a few hours. The first major algorithmic short-cut is restricting the search to ascending sequences with respect to coordinates in a basis and lexicographic ordering, thus omitting all permutations arising from the same sequence. The second short-cut is keeping track of all group elements not in the set of negative subsums in additional vectors—namely G_1, G_2, G_3, G_4, and G_5 in the SEA (see Algorithm 1). These vectors are used to massively speed up the Sumset Computation Algorithm (SCA) (see Algorithm 2) by avoiding many unnecessary tests. Typically, the vectors G_i, for $i \in [1, 5]$, consist of only a few hundred group elements while $\#G = 20000$—this means a speed up by a factor of about 20 to 200 in each step of the descending inner loops in the SCA (see Algorithm 2). As a last step of optimization, we pre-compute a look-up table for subtraction in G, which is stored in a very specific way such that we can use it for the tests in the SCA (see Algorithm 2)

Algorithm 1 Sequence Extension Algorithm: $(g_1, g_2, g_3, g_4, g_5, g_6) \leftarrow \mathrm{SEA}(S)$

```
σ0 ← −Σ(S) ∪ {0}
for all g1 ∈ G such that g1 ∉ σ0 do
  σ1 ← ∅, σ2 ← ∅, σ3 ← ∅, σ4 ← ∅, σ5 ← ∅
  G1 ← ∅, G2 ← ∅, G3 ← ∅, G4 ← ∅, G5 ← ∅
  σ1 ← σ0 ∪ (σ0 − g1)
  for all g ∈ G such that g ∉ σ1 do
    G1 ← G1 ∪ {g}
  end for
  for all g2 ∈ G1 such that g2 ≤ g1 do
    (G2, σ2) ← SCA(G1, σ1, g2)
    for all g3 ∈ G2 such that g3 ≤ g2 do
      (G3, σ3) ← SCA(G2, σ2, g3)
      for all g4 ∈ G3 such that g4 ≤ g3 do
        (G4, σ3) ← SCA(G3, σ3, g4)
        for all g5 ∈ G4 such that g5 ≤ g4 do
          (G5, σ4) ← SCA(G4, σ4, g5)
          for all g6 ∈ G5 such that g6 ≤ g5 do
            return (g1, g2, g3, g4, g5, g6)
          end for
        end for
      end for
    end for
  end for
end for
```

Algorithm 2 Sumset Computation Algorithm: $(G', \sigma') \leftarrow \mathrm{SCA}(G, \sigma, g)$

```
σ' ← σ
G' ← ∅
for all h ∈ G do
  if (g + h) ∈ σ then
    σ' ← σ' ∪ {h}
  else
    G' ← G' ∪ {h}
  end if
end for
return (G', σ')
```

and benefit from data caching and pre-fetching on modern CPUs while accessing the elements in a single line of the look-up table.

Test Set	Test Sequences	Complete	Hits	Extensions	Compute Time
a	81	27	0	0	28,366
b	81	52	5	92	26,670
c	81	52	5	252	26,688
d	81	75	4	196	15,808
	324	206	14	540	**97,534**

TABLE 1. Statistics of the four test runs a, b, c, and d on the *cineca* supercomputer. The compute time is given in hours w.r.t. a single IBM Power6 4.7 GHz SMT CPU core.

The computations for the test sequences a, b, c, and d on the *cineca* supercomputer used 64 threads with a single master and 63 slaves. The parallel efficiency of the algorithm, due to the independent nature of the computations, proved to be very good. The *cineca* supercomputer is an IBM pSeries 575 Infiniband

cluster with 168 computing nodes and 5376 computing cores. Every node has eight IBM Power6 4.7 GHz quad-core CPUs with simultaneous multi-threading (SMT) and 128 GB of shared memory. Performance tests of the parallel algorithm showed that the best configuration to run a single work item on is a single computing node with 64 threads with SMT enabled. Single node work loads were also scheduled typically within a day on the *cineca* supercomputer. The complete run of all test sets a, b, c, and d took about a week on the *cineca* supercomputer with an equivalent of nearly 100,000 SMT CPU core hours computation time. The run time of a work item on a single computing node was limited to six hours wall clock time by batch processing system policy. Nevertheless, most work items finished within these time restrictions, namely, 206 out of 324, and the ones that did not finish had most of the time only very few elements left to check, so we decided not to reschedule these work items for completion. The full statistics of the computations is given in Table 1.

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